Sheaves on Proj

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Let $R = \bigoplus_{j \in \mathbb{N}} R_j$ be an $\mathbb{N}$-graded ring. We constructed a scheme $(X, \mathcal{O}_X) = (\operatorname{Proj}(R), \mathcal{O}_{\operatorname{Proj}(R)})$; Set $R_+ = \bigoplus_{j>0} R_j$

Recall $\operatorname{Proj}(R) = \{ p \mid p \text{ homogeneous prime}, \ p \not\supseteq R_+ \}$

The top on $\operatorname{Proj}(R)$ has open basis $\{D_+(f)\}$ f homo of +ve deg

$$D_+(f) = \{ p \in \operatorname{Proj}(R) \mid f \notin p \}$$

$$\mathcal{O}_X (D_+(f)) \cong (R[1/f])_0 .$$

§ Recall, a $\mathbb{Z}$-graded R mod is $M \in \mathrm{Mod}_R$ such that $M = \bigoplus_{\lambda \in \mathbb{Z}} M_\lambda$ as an ab group

and $R_{\lambda_1} \cdot M_{\lambda_2} \subseteq M_{\lambda_1 + \lambda_2}$, $\forall \lambda_1 \in \mathbb{N}, \lambda_2 \in \mathbb{Z}$.

- A graded R-lin map (or simply a graded map) between two graded mods is an R lin map $\varphi: M \to N$ s.t $\varphi(M_\lambda) \subseteq N_\lambda$ $\forall \lambda \in \mathbb{Z}$.
- The cat of $\mathbb{Z}$-graded R-mods with graded R-lin map forms a cat denoted Mod_R^{gr}

Given $M \in \mathrm{Mod}_R^{gr}$, we construct a sheaf of $\mathcal{O}_X$-mods, denoted $\tilde{M}$ on $X = \operatorname{Proj} R$

Caution: This $\tilde{M}$ is not the qcoh sheaf $\tilde{M}$ on $\operatorname{Spec}(A)$.

- Note given a mult closed set S of homogeneous elts of R, $S^{-1}R$ is $\mathbb{Z}$-graded ring. $S^{-1}M$ is a $\mathbb{Z}$-graded mod / $S^{-1}R$.

$$(S^{-1}R)_\lambda = \{ r/s \in S^{-1}R \mid r, s \text{ homo}, \ \deg r - \deg s = \lambda \}$$

$$(S^{-1}M)_\lambda = \{ m/s \in S^{-1}M \mid m, s \text{ homo}, \ \deg m - \deg s = \lambda \}$$

- For $p \in \operatorname{Proj}(R)$, set S_p = homo elts of $R \setminus p$

$$M_{(p)} = (S_p^{-1} M)_0$$

Def of $\tilde{M}$: Consider the presheaf of $\mathcal{O}_X$-mods on X;

$$\tilde{M}(U) = \Big\{ \text{set maps } s: U \to \coprod_{p \in U} M_{(p)} \ \Big| \ \begin{array}{l} \bullet\ s(p) \in M_{(p)} \\ \bullet\ \text{For every } U, \exists \text{ a open nbhd } V \subseteq U \text{ of} \\ p, \text{ a homo } f \notin \bigcup_{q \in V} q \\ \& \ m \in M \text{ s.t} \\ s(q) = m/f \in M_{(q)} \end{array} \Big\}$$

It's implicit that $\deg m = \deg f$

Recall: $\tilde{R} = \mathcal{O}_X$, So for $s \in \mathcal{O}_X(U)$, $t \in \tilde{M}(U)$

$$(s\,t)(q) = \underset{R_{(q)}}{\underset{\uparrow}{s(q)}} \cdot \underset{M_{(q)}}{\underset{\uparrow}{t(q)}} \in M_{(q)} \quad \forall q \in U$$

Thm: 1) $\tilde{M}$ is a sheaf for any $M \in \mathrm{Mod}_R^{gr}$.

For $p \in \operatorname{Proj} R$, the natural map $M_{(p)} \to (\tilde{M})_p$ is an isom with inverse given by $s \mapsto s(p)$ for a section $s \in \tilde{M}(U)$.

$p \in U$.

2) Given a graded R-lin map $\varphi : M \to N$, naturally have $\tilde{\varphi} : \tilde{M} \to \tilde{N}$

for $t \in \tilde{M}(U)$, $p \in U$ $\tilde{\varphi}_U(t)(p) = \varphi_{(p)}(t(p))$; $\varphi_{(p)} : M_{(p)} \to N_{(p)}$

3) $\widetilde{\varphi_1 \cdot \varphi_2} = \tilde{\varphi}_1 \cdot \tilde{\varphi}_2$, $\widetilde{id_M} = id_{\tilde{M}}$

4) So $M \to \tilde{M}$ is gives a functor $Mod_R^{gr} \to Mod_{\mathcal{O}_X}$.

Thm: For $M \in Mod_R^{gr}$; $\tilde{M} \in Qcoh(X)$.

Moreover $\Gamma(D_+(f), \tilde{M}) \cong (M[1/f])_0$ for every homo elt f of +ve deg.

Pf. Since being qcoh is a local property, it's enough to check that for any homo elt f of of +ve deg $\tilde{M}|_{D_+(f)}$ is qcoh.

Set $\Gamma = M[1/f]_0$. The identity map $\Gamma \to \Gamma(D_+(f), \tilde{M})$ gives an $\mathcal{O}_{D_+(f)}$-line map

α $\tilde{\Gamma} \longrightarrow \tilde{M}|_{D_+(f)}$ as $\tilde{\Gamma}$ is qcoh on the affine $D_+(f)$.

Claim. α is an isom.

Pf. We check isom at the stalks at $p \in D_+(f)$.

Recall that $D_+(f) \xrightarrow{\sim} Spec(R[1/f]_0)$
$$q \longmapsto q R[1/f] \cap R[1/f]_0 = q_a$$

The map induced by $\alpha : (M[1/f])_{0\,p_a} \cong \tilde{\Gamma}_{p_a} \longrightarrow \tilde{M}_p \xrightarrow{\sim} M_{(p)}$ α_x

$$\begin{tikzcd} \tilde{\Gamma}_{p_a} \arrow[r] & \tilde{M}_p \\ \frac{m/f^a}{g/f^{e_2}} \arrow[u, phantom, "\in"] \arrow[r, mapsto] & \frac{m f^{e_2}}{g f^{e_1}} \arrow[u, phantom, "\in"] \end{tikzcd}$$

$g \in S_p$,

Note $(M[1/f]_0)_{p_a} = [(S_p)^{-1} M[1/f]]_0 = (S_p^{-1} M)_0$

So ~~B~~ α α_x is an isom.

Rmk. So $\tilde{M}$ can be thought to be obtained by gluing $\widetilde{M[1/f]_0}$ on $D_+(f)$ for different f's.

Thm. 1) $f : M \to N$ be a graded R-mod map.
$\ker \tilde{f} \xleftarrow{\sim} \widetilde{\ker f}$, $\operatorname{coker} \tilde{f} \xrightarrow{\sim} \widetilde{\operatorname{coker} f}$.

2) If $0 \to M' \to M \to M'' \to 0$ is exact in Mod_R^{gr}
[This includes the hypothesis that all the maps are graded]
Then $0 \to \tilde{M}' \to \tilde{M} \to \tilde{M}'' \to 0$ is also exact in $Mod_{\mathcal{O}_X}$

3) $(\bigoplus_{i \in I} M_i)^{\sim} \xleftarrow[\text{in } Mod_{\mathcal{O}_X}]{\sim} \bigoplus_{i \in I} \tilde{M}_i$ for any set I.

Thm. 1) $\tilde{M} = 0 \iff \forall f$ homo of +ve deg $M[1/f]_0 = 0$
$\iff$ for a collection of homo elt of +ve deg $f_1, f_2 \ldots, f_r$ s.t $X = \bigcup_{i=1}^{r} D_+(f_i)$ and
and any $m \in M_n$ where $\deg f_i \mid n \ \forall i$,
$\exists\ n_1, n_2 \ldots n_r$ s.t $f_i^{n_i} \cdot m = 0$

[A special case is when one can choose f_i's of deg 1 s.t $X = \bigcup_{i=1}^{r} D(f_i)$]

2) For $M \in \mathrm{Mod}_R^{gr}$,

$i : \bigoplus_{\lambda \in \mathbb{N}} M_\lambda \hookrightarrow M$. $\tilde{i}$ is an isom.

Pf. $\tilde{M} = 0 \iff \forall f$ homo of +ve deg $\tilde{M}|_{D_+(f)} = 0$

$\underset{\tilde{M}\text{ q.coh}}{\iff}$ ——" —— $M[1/f]_0 = 0$

$\iff$ for a collection $\{f_i\}_{i \in I}$, f_i homo of +ve deg s.t $X = D_+(f_i)$, $M[1/f_i]_0 = 0$

Now assume $X = \bigcup_{i=1}^{r} D(f_i)$

$M[1/f_i]_0 = \{ m/f_i^n \mid \deg m = n \deg f_i \}$

$M[1/f_i]_0 = 0 \iff$ for $m \in M_\lambda$ s.t $\deg f_i \mid \lambda$

$m/f_i^{\lambda/\deg f_i} = 0 \in M[1/f_i]_0 \subseteq M[1/f_i]$

$\iff f_i^{n_i} m = 0$ for some n_i.

2) Consider the exact seq in Mod_R^{gr}

$$0 \to \bigoplus_{\lambda \in \mathbb{N}} M_\lambda \to \overset{M}{\overset{\shortparallel}{\bigoplus_{\lambda \in \mathbb{Z}} M_\lambda}} \to Q \to 0$$

Every nonzero ~~dt~~ homo elt in Q lifts to a non-zero homo elt, m, of $-$ve deg in M

Now for every homo elt, f, of R of +ve deg

$\deg f^n \cdot m \geq 0$ for $n >> 0 \Rightarrow f^n \cdot \underset{\text{im in } Q}{\overset{\uparrow}{\bar{m}}} = 0$

$\Rightarrow \tilde{Q} = 0$

$\Rightarrow 0 \to \widetilde{\bigoplus_{\lambda \in \mathbb{N}} M_\lambda} \to \tilde{M} \to 0$ is exact.

End of 08.11.24 Lec

§. $\mathcal{O}_X(n)$, $\mathcal{F}(n)$, . . $X = (\mathrm{Proj}(R), \mathcal{O}_{\mathrm{Proj}(R)})$

Def. • For $M \in \mathrm{Mod}_R^{gr}$, $M(n) \in \mathrm{Mod}_R^{gr}$ ~~is~~ the object whose underlying R-mod is M, but

$$M(n)_m = M_{n+m}$$

• $\mathcal{O}_X(n) := \widetilde{R(n)}$ • For $\mathcal{F} \in \mathrm{Mod}_{\mathcal{O}_X}$, $\mathcal{F}(n) := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$

$\mathcal{F}(n)$ is called the n-th Serre twist of $\mathcal{F}$.

Note There is a map $M_0 \to \Gamma(X, \tilde{M})$ and so

$M_n = (M(n))_0 \longrightarrow \Gamma(X, \widetilde{M(n)})$

Def: Y scheme, $\mathcal{G} \in \mathrm{Mod}_{\mathcal{O}_Y}$ is called to be free if $\exists$ an open covering $Y = \bigcup_{\alpha \in I} U_\alpha$ s.t $\mathcal{F}|_{U_\alpha} \xrightarrow{\sim} \bigoplus_{r_\alpha} \mathcal{O}_{U_\alpha}$, $r_\alpha \in \mathbb{N}$ in $\mathrm{Mod}_{\mathcal{O}_{U_\alpha}}$

Prop • There are natural maps $\mathcal{O}_X(n) \otimes \mathcal{O}_X(m) \to \mathcal{O}_X(n+m)$ and $\mathcal{F}(n) \otimes \mathcal{O}_X(m) \to \mathcal{F}(n+m)$.

• Suppose $\deg f = m > 0$, Then

(i) $\mathcal{O}_{D_+(f)} \longrightarrow \mathcal{O}_X(m)|_{D_+(f)}$ is an isom in $\mathrm{Mod}_{\mathcal{O}_{D_+(f)}}$

$1 \longmapsto f/1$

(ii) $\mathcal{F}(n)|_{D_+(f)} \otimes \mathcal{O}_X(m)|_{D_+(f)} \xrightarrow{\sim} \mathcal{O}_X(n+m)|_{D_+(f)}$

__Pf.__ (i) Using qcoh, enough to prove

$$\begin{tikzcd} \Gamma(D_+(f),\mathcal{O}_X) \arrow[r] & \Gamma(D_+(f),\mathcal{O}_X(m)) \\ R[1/f]_0 \arrow[u, "\cong"] \arrow[r] & {[R(m)[1/f]]_0 = (R[1/f])_m} \arrow[u, "\wr"'] \\ 1 \arrow[r, mapsto] & f/1 \\ s/f & s \arrow[l, mapsto] \end{tikzcd}$$

is an isom

inverse

(ii) Since $\mathcal{F}(n) := \mathcal{F} \otimes \mathcal{O}_X(n)$, enough to prove

$\mathcal{O}_X(n)|_{D_+(f)} \otimes \mathcal{O}_X(m)|_{D_+(f)} \longrightarrow \mathcal{O}_X(n+m)|_{D_+(f)}$

is an isom

q-coh $\Rightarrow$ d enough to check the isom $x \otimes y \longmapsto xy$

$$\begin{tikzcd} R[1/f]_n \otimes_{R[1/f]_0} R[1/f]_m \arrow[r] \arrow[d, "\wr"] & R[1/f]_{n+m} \arrow[d, equal] \\ R[1/f]_n \otimes_{R[1/f]_0} f\cdot R[1/f]_0 \arrow[r, "\sim"] & f\cdot R[1/f]_n \end{tikzcd}$$

§ Given $\mathcal{F} \in QCoh(X)$, $X = Proj(R)$ we want to construct a __candidate__ mod M s.t $\tilde{M} \xrightarrow{\sim} \mathcal{F}$.

__Caution.__ Such an M need not exist.

__Def.__ Given $\mathcal{F} \in Mod_{\mathcal{O}_X}$, set $\Gamma_*\mathcal{F} = \bigoplus_{n\in\mathbb{Z}} \Gamma(X,\mathcal{F}(n))$

$\underline{\Gamma_*\mathcal{F}} = \bigoplus_{n\in\mathbb{Z}} \mathcal{F}(n)$

__Prop__
- $\underline{\Gamma_*\mathcal{F}}$ is naturally a $\underline{\Gamma_*\mathcal{O}_X}$ mod
- There is a map of graded rings $R \to \Gamma_*\mathcal{O}_X$.
- Thus $\Gamma_*\mathcal{F}$ is naturally an R-mod.
- There is an $\mathcal{O}_X$-lin map $\widetilde{\Gamma_*\mathcal{F}} \longrightarrow \mathcal{F}$

__Pf.__ • $\Gamma_*\mathcal{O}_X = \bigoplus_{n\in\mathbb{N}} \Gamma(X,\mathcal{O}_X(n))$, have

$$\begin{tikzcd} \Gamma_*(X,\mathcal{O}_X(m)) \times \Gamma_*(X,\mathcal{O}_X(n)) \arrow[d] \\ \Gamma(X,\mathcal{O}_X(n+m)) \end{tikzcd}$$

making $\Gamma_*\mathcal{O}_X$ a ring.

Hence
$$\begin{array}{ccccc} \Gamma(X,\mathcal{F}(n)) & \times & \Gamma(X,\mathcal{O}_X(m)) & \longrightarrow & \Gamma(X,\mathcal{F}(n+m)). \\ \| & & \| & & \| \\ \Gamma_*\mathcal{F}_n & & \Gamma_*\mathcal{O}_{X,m} & & \Gamma_*\mathcal{F}_{m+n} \end{array}$$

- $$\begin{tikzcd} R_n \arrow[r] & \Gamma(X,\mathcal{O}_X(n)) \\ r \arrow[r, mapsto] & r/1 \end{tikzcd}$$

- We describe $\widetilde{\Gamma_* \mathcal{F}}(D_+(f)) \longrightarrow \mathcal{F}(D_+(f))$

qcoh $\Rightarrow$ enough to produce $\Gamma_* \mathcal{F}[1/f]_0 \longrightarrow \Gamma(D_+(f), \mathcal{F})$

Note $\Gamma_* \mathcal{F}|_{D_+(f)} = \bigoplus_{\lambda \in \mathbb{Z}} \Gamma(D_+(f), \mathcal{F}(\lambda))$ is

nat a $\Gamma_* \mathcal{O}_{D_+(f)} = \bigoplus_{\lambda \in \mathbb{Z}} \Gamma(D_+(f), \mathcal{O}_X(m))$ mod.

In the later ring the image of f via

$$\begin{tikzcd} R \arrow[r] \arrow[dr] & \Gamma_* \mathcal{O}_X \arrow[d] \\ & \Gamma_* \mathcal{O}_{D_+(f)} \end{tikzcd}$$

is invertible as $1/f \in \Gamma(D_+(f), \mathcal{O}(-\deg f))$

Thus have a graded $R[1/f] \longrightarrow \Gamma_* \mathcal{O}_{D_+(f)}$. This makes $\Gamma_* \mathcal{F}|_{D_+(f)}$ an $R[1/f]$ module and induces a graded R-lin map, $\Gamma_* \mathcal{F}[1/f] \longrightarrow \Gamma_* \mathcal{F}|_{D_+(f)}$

Restrict to the deg 0 piece to obtain the desired map.

Thm Assume R is a finitely generated R_0 alg; i.e $\exists$ $g_1, g_2, \dots, g_r$ +ve deg homo elts s.t

$R_0[x_1, \dots x_r] \longrightarrow R$, R_0-lin is sur.

$x_i \longmapsto g_i$

- For $\mathcal{F} \in \mathrm{Qcoh}(X)$, the map $\widetilde{\Gamma_* \mathcal{F}} \longrightarrow \mathcal{F}$ is an isom.

Pf: Our hypothesis $\Rightarrow$ $R_+ \underset{\text{ideal}}{=} (g_1, g_2, \dots, g_r)$

$\Rightarrow X = \bigcup_{i=1}^{r} D_+(g_i)$

Replace g_i by $g_i^{\prod_{j \neq i} \deg g_j}$ to assume $\deg g_i$ are all equal $= d$. Note $X = \bigcup_{i=1}^{r} D_+(g_i)$.

Note $\mathcal{O}_X(nd)$ is invertible for $\forall n$.

Since $\mathcal{F}$ is qcoh it's enough to check that the induced map $(\Gamma_* \mathcal{F}[1/g_i])_0 \longrightarrow \Gamma(D_+(g_i), \mathcal{F})$ $(\dagger)$ is an isom $\forall i$

Needed lemma

Lemma: Y quasi compact scheme, $\mathcal{G} \in \mathrm{Qcoh}(Y)$, $\mathcal{L}$ invertible, $s \in \Gamma(Y, \mathcal{L})$. Let

$D_s = \{ y \in Y \mid s_y \notin m_y \mathcal{L}_y \}$ m_y is the max ideal of $\mathcal{O}_{Y,y}$.

(i) D_s is open

(ii) Given $t_1 \in \Gamma(Y, \mathcal{G})$ if $t_1|_{D_s} = 0$,
$\exists$ $n \in \mathbb{N}$ s.t $t_1 \otimes s^n = 0 \in \Gamma(Y, \mathcal{G} \otimes \mathcal{L}^n)$

(iii) Assume intersection of every two open affines have a finite covering by affine opens in Y. [This is immediate if Y is noetherian or Y is (quasi) separated]

Given $t \in \Gamma(D_s, \mathcal{G})$, $\exists$ $n \in \mathbb{N}$, $\tilde{t} \in \Gamma(Y, \mathcal{G} \otimes \mathcal{L}^n)$
s.t $\underset{\Gamma(D_s, \mathcal{G} \otimes \mathcal{L}^n)}{\underset{\cap}{t \otimes s^n}} = \tilde{t}|_{D_s}$

Pf. (i) Quasicompactness is unnecessary. Choose an affine open covering $Y = \bigcup_j U_j$ s.t
$\mathcal{O}_{U_j} \xrightarrow{\sim} \mathcal{L}|_{U_j}$ $\quad 1 \longrightarrow s_j$, $\quad s|_{U_j} = f_j s_j$, $f_j \in \Gamma(U_j, \mathcal{O}_{U_j})$

$D_s \cap U_j = D(f_j) \underset{\text{open}}{\subseteq} U_j$. Thus $D_s \cap U_j$ is open in U_j. $\forall j$. Thus D_s is open in X. ∎

Since Y is quasi-compact, we choose a finite subcover of U_j $\quad Y = \bigcup_{j=1}^{r} U_j$. Fix this for (ii), (iii)

(ii) $t_1|_{U_j \cap D_s} = 0$, $U_j \cap D_s = D_{U_j}(f_j)$ [This means the basic affine open given by f_j inside U_j]

Since U_j is affine and $\mathcal{G}|_{U_j}$ is qcoh $\quad f_j^{n_j} \cdot t_1 = 0 \in \Gamma(U_j, \mathcal{G}|_{U_j})$ for some n_j
$\implies t_1 \otimes s^{n_j}|_{U_j} = 0 \in \Gamma(U_j, \mathcal{G} \otimes \mathcal{L}^{n_j})$
Take $n = \max\{n_1, \dots, n_r\}$, Then $t_1 \otimes s^n|_{U_j} = 0$ $\forall j$
$\implies t_1 \otimes s^n = 0 \in \Gamma(Y, \mathcal{G} \otimes \mathcal{L}^n)$

(iii) Since $D_s \cap U_j = D(f_j)$ in U_j and
$\mathcal{G}(D_s \cap U_j) = \mathcal{G}(U_j)[1/f_j]$ [$\because$ $\mathcal{G}$ is qcoh]
$\exists$ n_j s.t $f_j^{n_j} \cdot t$ is the restriction of a section in $\Gamma(\mathcal{G}, U_j)$ to $D(f_j) = D_s \cap U_j$. This means
$\exists$ $t_j \in \Gamma(U_j, \mathcal{G} \otimes \mathcal{L}^{n_j})$ s.t $t \otimes s^{n_j} = t_j|_{D_s \cap U_j}$

Let $n_0 = \max\{n_1, n_2 \ldots , n_r\}$

set $t'_j = t_j \otimes s^{n_0 - n_j} \in \Gamma(U_j, \mathcal{G} \otimes \mathcal{L}^{n_0})$

On $U_{j_1} \cap U_{j_2} \cap D_s$ $t'_{j_1} = t'_{j_2} = t|_{U_{j_1} \cap U_{j_2} \cap D_s} \otimes s^{n_0}|_{U_{j_1} \cap U_{j_2} \cap D_s}$

By our hypothesis $U_{j_1} \cap U_{j_2} = \bigcup_{\alpha \in I} V_\alpha^{j_1 j_2}$, where $|I| < \infty$.

Using (ii) on each $V_\alpha^{j_1 j_2}$ and finiteness of the covering $U_{j_1} \cap U_{j_2} = \bigcup V_\alpha^{j_1 j_2}$

find $n_{j_1 j_2}$ such that $t'_{j_1} \otimes s^{n_{j_1 j_2}} = t'_{j_2} \otimes s^{n_{j_1 j_2}}$
$\in \Gamma(U_{j_1} \cap U_{j_2}, \mathcal{G} \otimes \mathcal{L}^{n_0 + n_{j_1 j_2}})$

set $n = \max_{j_1, j_2 \in I} \{n_0 + n_{j_1 j_2}\}$

Then $t'_j \otimes s^n \in \Gamma(U_j, \mathcal{G} \otimes \mathcal{L}^n) \ \forall j$

and $t'_{j_1} \otimes s^n|_{U_{j_1} \cap U_{j_2}} = t'_{j_2} \otimes s^n|_{U_{j_1} \cap U_{j_2}} \ \forall j_1, j_2$

$\Rightarrow \tilde{t} \in \Gamma(Y, \mathcal{G} \otimes \mathcal{L}^n)$ s.t $\tilde{t}|_{U_j} = t_j \otimes s^n \ \forall j$

Clearly on $\tilde{t}|_{D_s} = t \otimes s^n$ on $\Gamma(D_s, \mathcal{G} \otimes \mathcal{L}^n)$

Back to (†)

injectivity. If s/g_i^n goes to zero

$s|_{D_+(g_i)} = 0 \Rightarrow \exists\, n_i$ s.t $s \cdot g_i^{n_i} = 0 \in \Gamma(X, \mathcal{F}(n_i d))$

$\Rightarrow s/g_i^n = 0$ in $(\Gamma_* \mathcal{F}[1/g_i])_0$

Surjectivity. Note $X = \bigcup D_+(g_i)$

$D_+(g_\alpha) \cap D_+(g_\beta) = D_+(g_\alpha g_\beta)$

So we can apply (ii) of Lemma above on X.

Given $t \in \mathcal{F}(D_+(g_i))$. $\exists\, n_i \in \mathbb{N}$ s.t

$t \otimes g_i^{n_i} = \tilde{t}|_{D_+(g_i)}$ for some $\tilde{t} \in \Gamma(X, \mathcal{F}(n_i d))$

Then $\tilde{t}/g_i^{n_i} \in (\Gamma_* \mathcal{F}[1/g_i])_0$ with its image

in $\mathcal{F}|_{D_+(g_i)}$ being t.